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VIBRATIONS AND BUCKLING OF AXIALLY LOADED STIFFENED CYLINDRICAL SHELLS WITH ELASTIC RESTRAINTS†

AVIV ROSEN and JOSEF SINGER

Department of Aeronautical Engineering, Technion–Israel Institute of Technology, Haifa, Israel

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Abstract—A theory is derived for calculation of the influence of elastic edge restraints on the vibrations and buckling of stiffened cylindrical shells. The stiffeners are considered "smeared" and the edge restraints can be axial, radial, circumferential or rotational. Extensive computations are performed for special kinds of stringer-stiffened shells, and the theoretical predictions are compared with experimental results. A method of definition of equivalent elastically restrained boundary conditions by use of vibration tests is discussed. Application of this technique to tests on 10 shells significantly reduces the scatter in the ratio of experimental to predicted buckling loads.

NOTATION

A_1, A_2	cross section of stringer or ring, respectively
b_1, b_2	stringer or ring spacing, respectively
$C4$	clamped boundary conditions ($u = v = w = w_{,x} = 0$)
c_1, c_2	stringer or ring width, respectively
d_1, d_2	stringer or ring height, respectively
E	Young's modulus
E_1, E_2	Young's modulus of stringer or ring, respectively
e_1, e_2	stringer or ring eccentricity, respectively, (distance from shell middle surface to stiffener centroid, positive when inward)
f	frequency
G	shear modulus of shell
G_1, G_2	shear modulus of stringer or ring, respectively
h	thickness of shell
I_{01}, I_{02}	moment of inertia with respect to shell middle surface of stringer or ring, respectively
I_{11}, I_{22}	moment of inertia of stringer or ring cross section about its centroidal axis, respectively
J_1, J_2	torsional rigidity of stringer or ring, respectively
$k_1{}^r$	$\dfrac{R(1-\nu^2)\gamma_1{}^r}{Eh}$ nondimensional elastic axial restraint
$k_2{}^r$	$\dfrac{R(1-\nu^2)\gamma_2{}^r}{Eh}$ nondimensional elastic circumferential restraint
$k_3{}^r$	$\dfrac{12(1-\nu^2)R^3\gamma_3{}^r}{Eh^3}$ nondimensional elastic radial restraint
$k_4{}^r$	$\dfrac{12(1-\nu^2)R\gamma_4{}^r}{Eh^3}$ nondimensional elastic rotational restraint
L	length of shell
$\bar{M}$	mass per unit area of stiffened shell
m	number of half longitudinal waves
$\bar{N}_x$	axial edge force per unit length of shell (positive when tension)
n	circumferential wave number
p	pressure on shell surface (positive in z direction)
P	axial compressive load
P_{cr}	theoretical axial buckling load
P_{sp}	calculated axial buckling load for shell with elastic axial or rotational restraint
P_{exp}	experimental axial buckling load
R	radius to shell middle surface
r	index which gets the value 1 or 2 at the boundaries $x_1 = \dfrac{-L}{2R}$; $x_2 = \dfrac{L}{2R}$, respectively
$SS3$	classical simple support ($w = M_x = N_x = v = 0$)
$SS4$	simple support ($w = M_x = u = v = 0$)
T	$h^2/12R^2$
t	time
U	strain energy
u^*, v^*, w^*	dimensional displacements
u, v, w	non-dimensional displacements $\left(\dfrac{u^*}{R}; \dfrac{v^*}{R}; \dfrac{w^*}{R}; \text{respectively}\right)$

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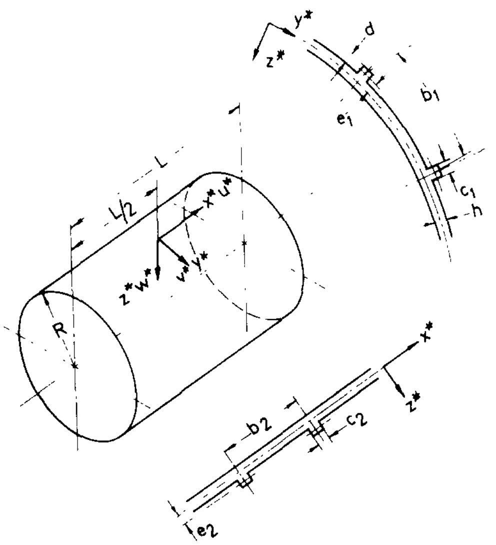

Fig. 1. Notation.

x^*, y^*, z^*	coordinates (see Fig. 1)
x, y, z	non-dimensional coordinates $\left(\frac{x^*}{R}; \frac{y^*}{R}; \frac{z^*}{R}\right.$; respectively$\left.\right)$
Z	Batdorf parameter $= \sqrt{(1-\nu^2)}L^2/Rh$
x_1, x_2, x_b	coordinates of shell edges
$\gamma_1{}^r$	elastic axial restraint for unit length
$\gamma_2{}^r$	elastic circumferential restraint for unit length
$\gamma_3{}^r$	elastic radial restraint for unit length
$\gamma_4{}^r$	elastic rotational restraint for unit length
η_{01}, η_{02}	$\frac{E_1 I_{01}}{b_1 D}; \frac{E_2 I_{02}}{b_2 D}$
η_{t1}, η_{t2}	$\frac{G_1 J_1}{b_1 D}; \frac{G_2 J_2}{b_2 D}$
θ	$[12(1-\nu^2)]^{1/4}[b_1(Rh)^{-1/2}/2\pi]$ Koiter's measure of total curvature [see 32]
μ_1, μ_2	$\frac{(1-\nu^2)E_1 A_1}{E b_1 h}; \frac{(1-\nu^2)E_2 A_2}{E b_2 h}$
ν	Poisson's ratio
ρ	"linearity", ratio between experimental buckling load and the theoretical value predicted by linear theory
ρ_{sp}	"linearity" with respect to elastic restraints
ω	angular velocity (frequency) $(2\pi f)$
$(\dot{\ })$	differentiation with respect to time
$(\ )_{,a}$	differentiation with respect to a
χ_1, χ_2	$\frac{(1-\nu^2)E_1 A_1 e_1}{E b_1 h R}, \frac{(1-\nu^2)E_2 A_2 e_2}{E b_2 h R}$
$\delta(\ )$	variation of the function in brackets.

1. INTRODUCTION

The influence of boundary conditions on the buckling of cylindrical shells has been studied extensively (see for example [1–10]). Radial, rotational and in-plane boundary conditions have been considered, mostly for axial compression loading and external pressure. Elastic restraints at the boundaries, which represent practical boundary conditions more truly, have also been studied by Almroth[8] and [13], Singer[11] and [12] and others. The elastic restraints usually represent the end rings or the connection of the shell to the other parts of the structure, though shells with elastic end rings have also been investigated by Cohen[14].

Boundary conditions have also a considerable influence on the vibrations of cylindrical shells, which has been discussed by many investigators (see for example Forsberg[15] or Nuckolls and Egle[16] who considered also elastic restraints). The vibrations of cylindrical shells with end rings were studied by El-Raheb and Babcock[17], who found that the end rings noticeably influence the frequencies and modes of vibrations, and that replacement of end rings by any of the natural or geometrical boundary conditions would involve considerable errors.

In stiffened shells, too, the influence of boundary conditions on buckling has been the subject of many investigations, including extensive Technion studies. In these studies (see for example [18–20]) the boundary conditions were found to have a large influence on the buckling of axially compressed stiffened cylindrical shells. Contrary to isotropic shells under axial compression, the in-plane boundary conditions were here found to be very important and the magnitude of their influence to depend strongly on shell and stiffener geometries. The different studies are summarized in a recent review[19].

Boundary conditions have also considerable influence on the vibrations of stiffened cylindrical shells, as demonstrated clearly by Sewall and Naumann[22]. They investigated the influence of different boundary conditions (simple supports—$SS3$, clamped–clamped, clamped–free and free–free) on the vibrations of isotropic and stringer-stiffened cylindrical shells (with internal or external stiffeners).

The vibrations of axially loaded stiffened cylindrical shells were recently investigated theoretically and experimentally by the authors[23–25]. A method of analysis was developed which yields the natural frequencies of a stiffened cylindrical shell under axial load and external pressure for any combination of natural and geometrical boundary conditions. The buckling axial load or critical pressure can also be found by this method. Good agreement with theoretical results was obtained in experiments on isotropic and stringer- or ring-stiffened shells.

The present study is an extension of that analysis to include also elastic restraints at the boundaries and a continuation of the experimental program. The correlation of the vibration tests with the theory is then employed for more precise definition of the boundary conditions, aiming at improved predictions of buckling loads.

2. EQUILIBRIUM EQUATIONS AND BOUNDARY CONDITIONS

The approach presented here is based on "smearing" of the stiffeners. This approach has been discussed in detail by Baruch and Singer[20] and by Singer, Baruch and Harari[21]. It was applied by Mikulas, McElman and Stein[26, 27], and later by other investigators—as well as the present authors (see [23–25])—to the vibration of stiffened shells. The accuracy of "smeared" theory as compared with more exact "discrete" stiffener analyses is also discussed in many of the studies reviewed in [23]. The more recent ones[28–31] show that the "smeared" approach yields accurate results, provided the stiffeners are uniformly spaced, fairly close and not too heavy. Since here the stiffeners obey these criteria, the use of "smeared" theory is justified. The rotational inertia of the shell cross-section is neglected, as it was shown (see [22, 28, 29]) that its effect is indeed negligible in stiffened shells.

The equilibrium equations and natural boundary conditions are derived by variation of the total potential of the shell, and the derivation was carried out according to both the Donnell theory and the more exact Flügge theory. The presentation, here, however, limits itself to Flügge's theory, and only points out in the last equations the terms which would not appear in a Donnell approach.

Following the development in [23], the first variational principle is applied to the total potential of the shell to which a new term U_{sp}, the elastic energy of the equivalent springs at the ends of the shell that represent the elastic restraints at the boundaries, is added. The stiffness of the end springs per unit length of the shell boundary is $\gamma_1{}^r$ in the axial direction, $\gamma_2{}^r$ in the circumferential direction, $\gamma_3{}^r$ in the radial direction and $\gamma_4{}^r$ for the rotational restraint, where the r index is 1 for the boundary x_1 and 2 for the boundary x_2. Hence one obtains

$$\delta U_{sp} = R^3 \int_0^{2\pi} \left[(-1)^r \gamma_1{}^r u\delta u + (-1)^r \gamma_2{}^r v\delta v + (-1)^r \gamma_3{}^r w\delta w + (-1)^r \frac{1}{R^2}\gamma_4{}^2 w_{,x}\delta w_{,x}\right]_{x=x_1}^{x=x_2} \mathrm{d}y. \quad (1)$$

The variation yields the equations of equilibrium and the boundary conditions which are presented, in terms of displacement, in detail in TAE Report 208[33] (see eqns (7) and (8) there). Note that eqns (7) of [33] are identical to eqns (A32) of [23], whereas the boundary conditions, eqns (8) of [33], differ slightly from eqns (A33) of [23] due to the elastic restraints.

If one assumes a membrane prestress state, and u, v, w represent the mode of vibration, one finally obtains the required three equations of equilibrium and boundary conditions presented in TAE Report 208[33] (eqns (9) and (10)) and given for convenience in the Appendix.

The equilibrium equations for a certain set of boundary conditions are solved by the method known as the "exact solution" which is described in [23] or [25]. By this method one can find the first, second, third and higher modes of frequencies of vibration of a stiffened cylindrical shell with any combination of elastic restraints at the boundaries, for a certain number of circumferential waves.

For the case of axial compression loading, considered here,

$$\bar{N}_x = -\frac{P}{2\pi R}. \tag{2}$$

The influence of increase of P on the vibrations of a stiffened cylindrical shell is studied. At a certain value of P the shell buckles. Dynamically, buckling is regarded as the state of vanishing frequency, which means that the shell departs from the state of equilibrium and does not return to it. Hence to obtain the buckling load, the value of P is sought for which the natural frequency vanishes. The buckling load is, therefore, determined in this manner, being obviously the lowest one at any possible value of the circumferential wave number.

A computer program has been developed to compute the results discussed in Section 4. The program was checked by comparison with results for no-load frequencies obtained in [22] and results for buckling loads obtained with the BOSOR 3 program. Details of some comparisons are given in [23].

3. EXPERIMENTAL PROCEDURE

The experimental method is similar to that employed earlier and described in detail in [24] or [25]. The axially loaded stiffened shells are vibrated by an acoustic driver inside the shell and resonance frequencies and mode shapes are detected by an outside scanning microphone. (The test set-up is shown in Fig. 2 of [33] or Fig. 3a of [24]).

As in earlier tests the integrally stiffened cylindrical shells were machined from 7075-T6 aluminum alloy extruded tubes by a process described in [32].

Three kinds of experimental boundary conditions were employed, as shown in Fig. 2. Types A and B are very similar and represent simple supports (free to rotate). Type C represents clamping. In this case the shell was placed in a circular groove of the same inner diameter, and the outside gap was filled with low melting point Cerroband. The type of boundary condition is indicated for each shell in Table 2.

4. RESULTS AND DISCUSSION

Ten stiffened shells were tested, which may be divided into four groups. Four shells, RO-16, 19, 44, 46, were "moderately" stiffened ($A_1/b_1h = 0.42$–0.47) and four shells, RO-15, 20, 43, 45, were "lightly" stiffened ($A_1/b_1h = 0.21$–0.23), with all 8 specimens having the same length $L/R = 1.25$. The remaining two shells were longer (with $L/R = 1.50$), one RO-18 being "moderately" stiffened and the other RO-17 "lightly" stiffened. The dimensions of the shells and stringers are given in Table 1. All the stringers have rectangular cross sections.

Only the group of "moderately" stiffened shells with $(L/R) = 1.25$ are discussed here. A detailed discussion of the other shells, as well as additional results for this group are presented in TAE Report 208[33]. However, results for the predicted and experimental buckling loads of all the shells are given in Table 2.

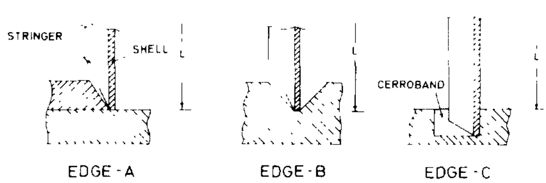

Fig. 2. Experimental boundary conditions.

Table 1. Dimensions and properties of shells

Shell	Type of stiffener	h [mm]	L [mm]	R/h	L/R	Z	d_1 [mm]	c_1 [mm]	b_1/h	A_1/b_1h	I_{11}/b_1h^3	$-e_1/h$	η_{t1}	θ
RO-15	Stringer	0.242	150	496	1.25	739	0.503	0.90	37.1	0.21	0.075	1.54	0.82	0.48
RO-20	Stringer	0.240	150	500	1.25	745	0.504	0.90	37.4	0.21	0.077	1.55	0.84	0.48
RO-43	Stringer	0.234	150	513	1.25	764	0.531	0.90	38.3	0.23	0.098	1.63	1.03	0.49
RO-45	Stringer	0.237	150	507	1.25	754	0.528	0.90	37.9	0.22	0.092	1.61	0.98	0.48
RO-17	Stringer	0.247	180	486	1.50	1042	0.498	0.90	36.3	0.20	0.069	1.51	0.75	0.47
RO-16	Stringer	0.239	150	503	1.25	748	1.054	0.95	37.5	0.47	0.757	2.71	4.77	0.48
RO-19	Stringer	0.242	150	496	1.25	739	1.003	0.90	37.1	0.42	0.595	2.57	3.77	0.48
RO-44	Stringer	0.231	150	520	1.25	774	1.030	0.90	38.8	0.45	0.713	2.73	4.54	0.49
RO-46	Stringer	0.234	150	513	1.25	764	1.030	0.90	38.3	0.44	0.713	2.70	4.36	0.49
RO-18	Stringer	0.243	180	494	1.50	1059	0.980	0.90	36.9	0.41	0.554	2.52	3.58	0.48

$R = 120.1$ mm; $b_1 = 8.97$ mm; $E = 7500$ kg/mm^2; $\nu = 0.3$.

Table 2. Experimental and theoretical buckling loads and modes of buckling (loads are given in kg)

Shell	A_1/b_1h	Exp. B.C.	P_{exp}	Exp. mode	$P_{cr}(SS3)$	$SS3$ mode	$P_{cr}(SS4)$	$SS4$ mode	$P_{cr}(C4)$	$C4$ mode	†	P_{sp}	Mode of P_{sp}	ρ	ρ_{sp}
RO-15	0.21	S.S(A)	1800	7/2	1927	10/1	2394	16/3	2452	15/2	12/1	2386	16/3	0.93	0.75
RO-20	0.21	S.S(A)	1930	8/2	1900	10/1	2366	16/3	—	—	11/1	2363	16/3	1.02	0.82
RO-43	0.23	S.S(B)	1775	8/2	1840	10/1	2337	16/3	—	—	13/1	2260	16/3	0.97	0.79
RO-45	0.22	CL(C)	2015	10/2	—	—	2376	16/3	2430	15/2	14/1	2390	16/3	0.83	0.84
RO-17	0.20	S.S(A)	1950	7/2	1968	9/1	2414	15/3	—	—	10/1	2408	15/3	0.99	0.81
RO-16	0.47	S.S(A)	3120	8/2	2586	10/1	3841	14/2	4213	12/1	12/1	3745	14/2	1.21	0.83
RO-19	0.42	S.S(A)	3010	8/2	2500	10/1	3646	14/2	—	—	10/1	3553	14/2	1.20	0.84
RO-44	0.45	S.S(B)	3005	9/2	2431	11/1	3528	14/2	—	—	10/1	3410	14/2	1.24	0.88
RO-46	0.44	CL(C)	2800	9/1	—	—	3580	14/2	3940	12/1	11/1	3820	14/2	0.71	0.73
RO-18	0.41	S.S(A)	3050	9/2	2360	9/1	3391	13/1	—	—	13/1	3437	13/1	1.29	0.89

†Mode for determination of the elastic restraints.

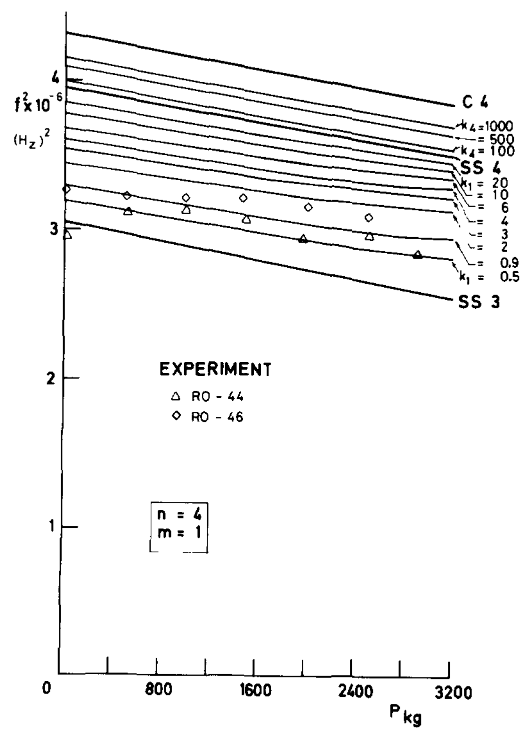

Fig. 3a. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

In Figs. 3a-f the influence of the axial load on the vibrations of shells with different elastic restraints is presented. As can be seen from Table 1 the dimensions of the shells RO-16, 19, 44, 46 are very close. Therefore, the initial calculations shown in Figs. 3 are for one shell only, RO-16. The influence of the elastic restraints is checked in two steps. First the axial restraint k_1 is varied between $SS3$ ($k_1 = 0$) and $SS4$ ($k_1 = \infty$) while rotation is unrestrained ($M_x = w = v = 0$). Then the

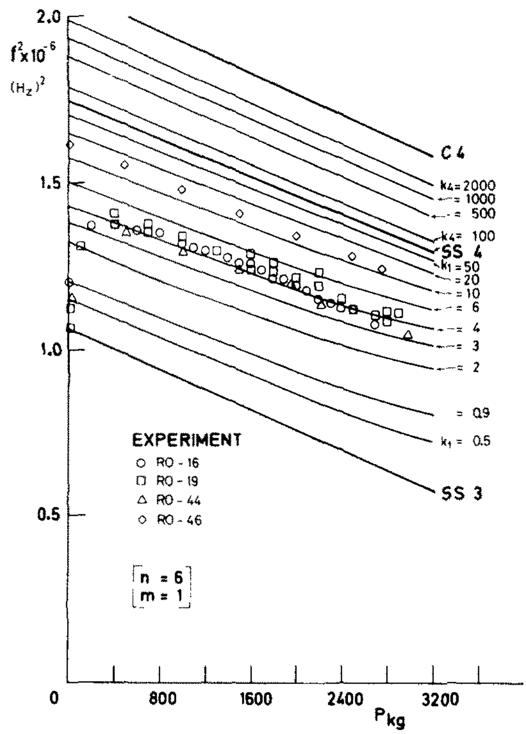

Fig. 3b. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

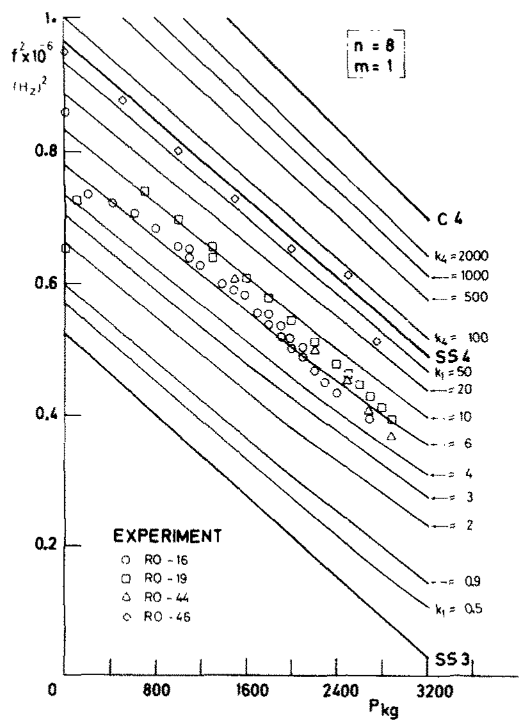

Fig. 3c. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

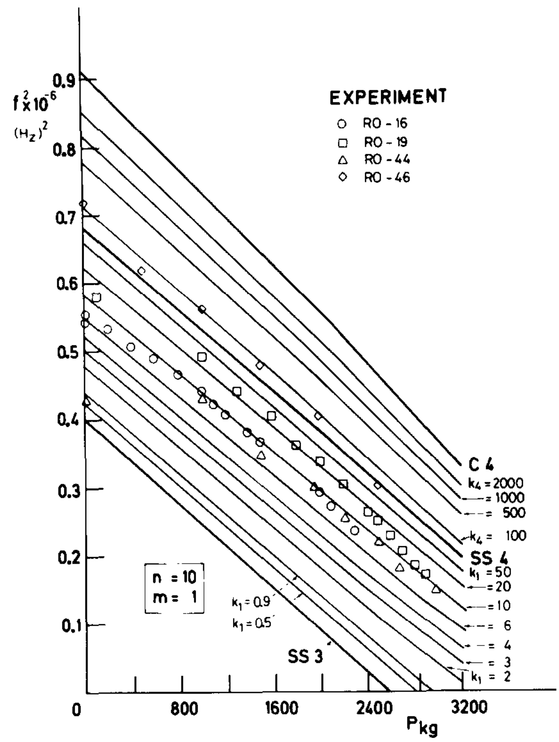

Fig. 3d. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

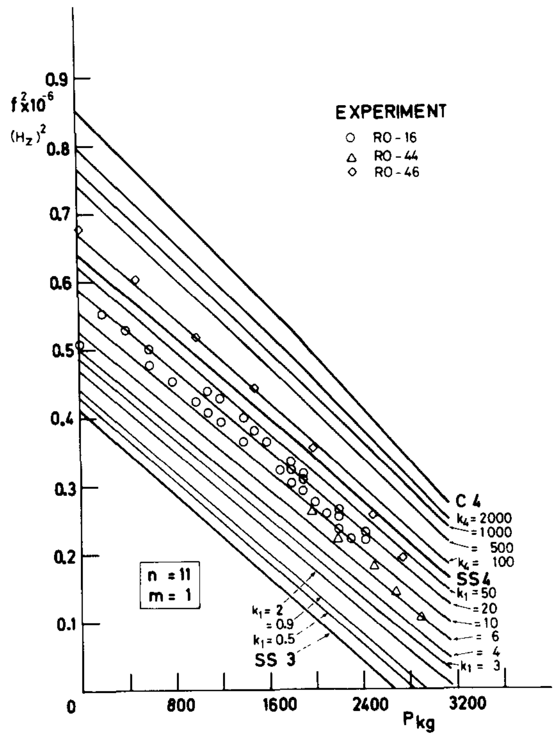

Fig. 3e. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

rotational restraint k_4 is varied between $SS4$ ($k_4 = 0$) and C_4 ($k_4 = \infty$) with edge displacements prevented ($u = v = w = 0$). Note that k_1 and k_4 have been chosen because they are the most significant elastic restraints for stringer-stiffened shells (radially, $w = 0$ or $k_3 \rightarrow \infty$ is assumed). Figures 3a–3f show the frequency for vibrations with one axial half wave and different circumferential wave numbers (n), between $n = 4$ and $n = 12$. As the circumferential wave

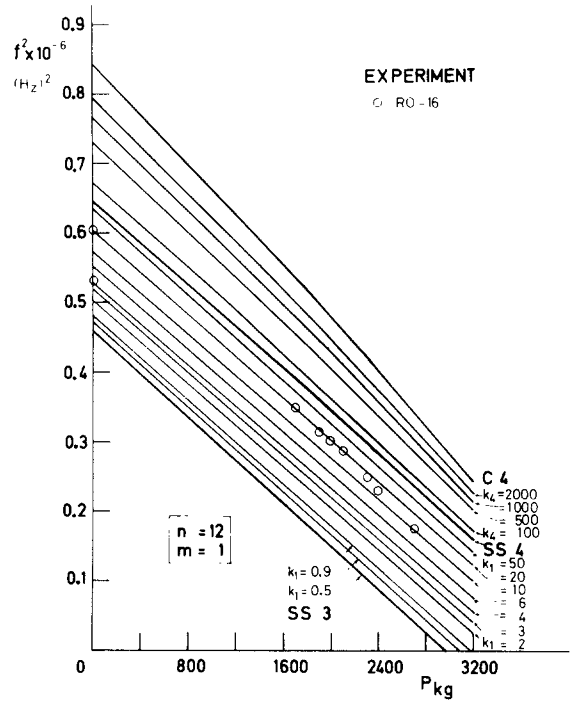

Fig. 3f. Frequency squared vs axial load in $m = 1$ mode, moderately stiffened shells ($L/R = 1.25$).

number increases the slope of the curve (variation of frequency squared with load) changes. In Figs. 3a–f the experimental results are also presented. Note that for certain circumferential wave numbers experimental results were obtained only for some of the shells. Of the shells in this group, RO-16, 19 and 44 were on simple supports whereas RO-46 was clamped. The experimental simple support allows local axial movement away from the loading plate but not in the opposite direction. Classical simple supports $SS3$ imply free local axial movement in both directions. The experimental boundary conditions are, therefore, stiffer than $SS3$, though they do not fully prevent axial movement as required by $SS4$ B.C's. Since the end plates are not entirely rigid, they may also be represented by springs, though—on account of their thickness and reinforcing ribs—fairly stiff springs. This boundary condition is, therefore, assumed to be equivalent to a certain elastic restraint k_1 between $SS3$ and $SS4$. Furthermore, consideration of the end plates as circular plates subjected to an axial line load with periodic variation around the circumference, yields an equivalent elastic restraint that increases rapidly with n. This may be one of the factors which contribute to the significant "n" dependence of the elastic restraints. For clamped ends it is also assumed that the clamping is not perfect and may be represented by an elastic rotational restraint k_4. As expected, in all cases the clamped shell RO-46 yields higher experimental results and those of the other shells fall in a narrow scatter band below.

One should remember, especially for simple supports, that at zero load the shell has not yet settled in its boundary conditions. A certain small load is required to achieve this and one has, therefore, to examine the results at zero load very carefully.

The experimental results in Figs. 3 indicate that the elastic restraints change with the circumferential wave number. For example, at a load of 1200 kg. for $n = 6$ (Fig. 3b) the simple supported shells yield $k_1 \approx 5$ and the clamped shell fits the prediction for $k_1 \approx 15$, whereas for $n = 10$ (Fig. 3d) $k_1 = 10$–20 for the simply supported shells and the clamped shell fits the prediction of $k_4 = 100$.

The change in the equivalent elastic restraints with circumferential wave number observed in the tests may be due to a combination of effects, in addition to the end-plate elasticity discussed, which can only be a partial factor. Preliminary studies on the influence of initial geometrical imperfections on the vibrations of isotropic cylindrical shells have shown a very significant influence both for axisymmetric [34] and asymmetric imperfections [35]. Hence, imperfections appear to be of similar importance as boundary conditions also for vibrations, and these conclusions may be expected to apply generally to stiffened shells as well. The influence of the initial imperfections was found to vary considerably with n (see [34] and [35]). Hence, the apparant change of equivalent elastic restraints with n may be a combined result of the actual change in boundary conditions with n and the change in the influence of initial imperfections with n.

The vibration tests at the different load levels are now employed for determination of the appropriate equivalent elastic restraints. These experimentally determined boundary conditions are then used in a linear buckling analysis. The details of the procedure for the 10 shells are presented in TAE Report 208[33]. Here only some examples are given.

In Fig. 4 the variation of the predicted frequency squared versus elastic restraint, extending from $SS3$ through $SS4$ to $C4$, is shown for a typical mode of vibration ($n = 12$, $m = 1$) at $P = 1200$ kg. In this case the small differences in dimensions between the four shells are taken into account and shown in the figure.

For each shell an experimental curve could be plotted in Figs. 3a–f, whenever enough experimental points are available, (they are not plotted in the figures for clarity). The elastic restraint appropriate to the observed frequency at $P = 1200$ kg. is found in Figs. 3a–f, as shown by the dotted lines. For example, in Fig. 4 the following values are obtained for $n = 11$: $k_1 = 20$ for RO-16 and $k_4 = 380$ for RO-46. The equivalent elastic restraints were found only when the available experimental results appeared consistent.

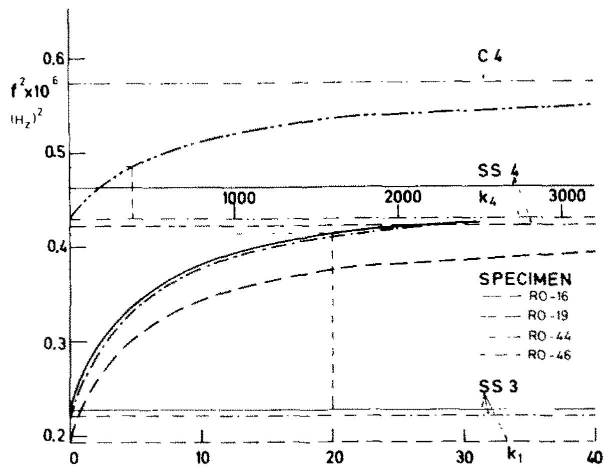

Fig. 4. Influence of elastic restraints on the vibrations (with $n = 11$, $m = 1$) of moderately stiffened shells ($L/R = 1.25$) at $P = 1200$ kg.

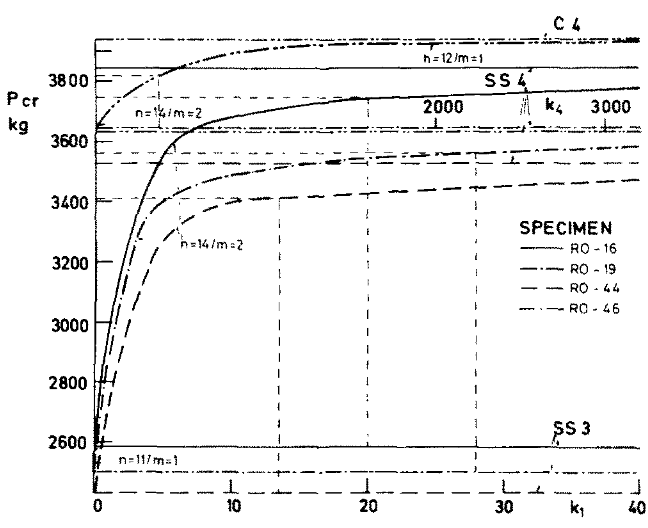

Fig. 5. Influence of elastic restraints on the buckling of moderately stiffened shells ($L/R = 1.25$).

Figure 5 shows the variation in the predicted buckling loads with elastic restraints. The small differences in the dimensions of the shells are taken into account and the respective modes of buckling appear on the curves. Note that throughout the range of restraints $n \geq 11$. This justifies the use of elastic restraints from vibrations with high circumferential wave number (like Fig. 4) in the calculations. Note also in Fig. 5 the changes in the buckling mode for minimum P_{cr} that occur as the elastic restraints vary.

With the values of k_1 obtained in Fig. 4 (and Figs. 6a and 6c of [33]) for the simply supported shells ($k_1 = 20$, 28, 13.5 for RO-16, 19, 44, respectively), one can find in Fig. 5 the predicted buckling loads for these elastic restraints (in the manner indicated by the dotted lines). The buckling loads are $P_{sp} = 3745$, 3553 and 3410 for shells RO-16, 19 and 44 respectively. Note also that all the points are in the region where the curve asymptotically approaches the $SS4$ value, and hence are much closer to $SS4$ boundary conditions than to $SS3$. For RO-46 the value of $k_4 = 380$ yields a buckling load of 3820 kg. The experimental buckling loads and mode shapes are given in Table 2. There is a scatter of 115 kg. in the buckling load for the simply supported shells (RO-16, 19 and 44), which is only about 4% of the buckling load. Note that these 3 shells are nominally identical, but in reality show up to 5% differences in geometric parameters. Surprisingly, however, the clamped shell of the group (RO-46), which would be expected to be stiffer, buckled at a lower load than the correspondingly simply supported shells. This will be discussed below.

In buckling experiments the term "linearity" ρ is often used to indicate the ratio between the experimental buckling load and the buckling load predicted by theory (see for example [19]). For simply supported shells the classical value predicted for $SS3$ boundary conditions is employed for reference and for clamped shells the value for $C4$ boundary conditions. This yields $\rho = 1.21$, 1.20 and 1.24 for RO-16, 19, 44 respectively and $\rho = 0.71$ for RO-46 (see Table 2), and there is a scatter of 53% in ρ. If, however, the "corrected" theoretical buckling loads P_{sp} are used for comparison, $\rho_{sp} = 0.83$, 0.85, 0.88 for RO-16, 19, 44 respectively and $\rho_{sp} = 0.73$ for RO-46, and the scatter reduces to 16%. With the corrected reference values, the "linearity" is always smaller than one, as would be expected when the boundary conditions are properly accounted for. These reductions in "linearity" can be mainly attributed to initial geometrical imperfections. Other imperfections, like eccentric and nonuniform loading, residual stresses and nonlinear material behavior also affect ρ (see also [36, 37]). The relatively smaller value of "linearity" for the clamped shell may be due to additional initial geometrical imperfections resulting from the particular experimental clamped boundary conditions used. Further studies are needed to clarify this degrading effect of clamping.

5. CONCLUSIONS

It has been shown that boundary conditions have a very significant influence on the vibrations and buckling of stringer-stiffened cylindrical shells. This influence depends strongly on the geometry of the shell and the stringers. Equivalent elastic restraints are defined to describe intermediate positions between the conventional simple support and clamped boundary conditions $SS3$, $SS4$ and $C4$. These elastic restraints represent more realistic boundary conditions in practice when the shells are supported on end rings, or other connecting elements.

Experimental results have been obtained for 10 specimens representing 4 different kinds of shells. In the tests, vibrations of the loaded shells on different experimental boundary conditions are compared with theoretical predictions and from the comparison the equivalent elastic restraints, which represent the real boundary conditions, are determined. The buckling loads for these equivalent elastic restraints are then computed.

The scatter between experimental buckling loads and theoretical predictions may be reduced when these experimentally defined boundary conditions are employed. The ratio of experimental buckling load to the predicted one, called "linearity", is introduced. With the usual $SS3$ boundary conditions for experimentally simply supported shells and $C4$ for experimentally clamped ones, one obtains for the 10 test shells a scatter of 0.71–1.29 in the "linearity" ρ. With theoretical predictions corresponding to the elastic restraints obtained from vibrations, one obtains a reduced scatter of only 0.73–0.89 in ρ_{sp}. This represents a reduction in scatter from 58 to 17%.

The experimental results show that the equivalent elastic restraints depend on the mode of vibration and vary significantly with the number of circumferential waves. However, this apparent change may be a combined effect of actual change in the boundary conditions with n

and a change in the influence of initial imperfections (geometrical ones, residual stresses, non-linear material behavior, etc.) with n. Hence one may try to define in future work an "overall equivalent elastic restraint" that will include both influences.

As a general conclusion from the experiments one can summarize that the experimental simple support conditions are usually closer to $SS4$ than to $SS3$ boundary conditions, and the clamped ones are somewhere between $SS4$ and $C4$ boundary conditions.

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REFERENCES

1. H. Ohira, Local buckling theory of axially compressed cylinders. *Proc. 11th Japan Natl Cong. Appl. Mech.* pp. 37–40 (1961).
2. H. Ohira, Boundary value problems of axially compressed circular cylinders. *Symp. Structures and Strength, Japan Society for Aero. and Space Sci.* (Feb., 1962).
3. N. J. Hoff, Buckling of thin shells. *Proc. Aerospace Scientific Symposium of Distinguished Lectures: in honour of Theodore von Kármán on his 80th Anniversary.* Institute of the Aerospace Science, New York (1961).
4. W. Nachbar and N. J. Hoff, On edge buckling of axially compressed circular cylindrical shells. *Quart. Appl. Math.* **20**, 267 (1962).
5. N. J. Hoff and L. W. Rehfield, Buckling of axially compressed circular cylindrical shells at stresses smaller than the classical critical value. *J. Appl. Mech., ASME,* **32**, 542 (1965); Stanford Univ. Dept. of Aero and Astro. *SUDAER Report No.* 191 (May 1964).
6. N. J. Hoff and T. C. Soong, Buckling of circular cylindrical shells in axial compression. *Int. J. of Mech. Sci.* **7**, 489 (1965).
7. N. J. Hoff, Low buckling stresses of axially compressed circular cylindrical shells of finite length. *J. Appl. Mech. ASME* **32**, 533 (1965).
8. B. O. Almroth, Influence of edge conditions on the stability of axially compressed cylindrical shells. *NASA CR*-161 (1965).
9. S. Kobayashi, The influence of the boundary conditions on the buckling load of cylindrical shells under axial compression. *NASA CR*-558 (1966).
10. L. Sobel, Effects of boundary conditions on the stability of cylinders subject to lateral and axial pressure. *AIAA J.* **2**, 1437 (1964).
11. J. Singer, The effect of axial constraint on the instability of thin circular cylindrical shells under external pressure. *J. of Appl. Mech., ASME* **27**, 737 (1960).
12. J. Singer, The effect of axial constraint on the instability of thin circular cylindrical shells under uniform axial compression. *Int. J. Mech., Sci.* **4**, 253 (1962).
13. B. O. Almroth, Influence of imperfections and edge restraint on the buckling of axially compressed cylinders. *NASA CR*-432 (1966).
14. G. A. Cohen, Buckling of axially compressed cylindrical shells with ring stiffened edges. *AIAA J.* **4**, 1859 (1966).
15. K. Forsberg, Influence of boundary conditions on the model characteristics of thin cylindrical shells. *AIAA J.* **2**(12), 2150 (1964).
16. C. E. Nuckolls and D. M. Egle, The influence of elastic boundary conditions on the resonant response of thin cylindrical shells. *AIAA Paper* 71–335 (1971).
17. M. El Raheb and C. D. Babcock, The effect of end rings on the natural frequencies of cylindrical shells. *Report SM* 71-1. Graduate Aeronautical Laboratories, California Institute of Technology (Jan. 1971).
18. T. Weller, M. Baruch and J. Singer, Influence of in-plane boundary conditions on buckling under axial compression of ring stiffened cylindrical shells. *Israel J. of Tech.* **9**, 397 (1971); also *TAE Report No.* 101, Technion Research and Development Foundation, Haifa, Israel (Oct. 1970).
19. J. Singer, Buckling of integrally stiffened cylindrical shells—a review of experiment and theory. *Contribution to the Theory of Structures*, pp. 325–357. Delft University Press, Delft (1972).
20. M. Baruch and J. Singer, Effect of eccentricity of stiffeners on the general instability of stiffened cylindrical shells under hydrostatic pressure. *J. of Mech. Eng. Sci.* **5**, 23 (1963).
21. J. Singer, M. Baruch and O. Harari, On the stability of eccentrically stiffened cylindrical shells under axial compression. *Int. J. Solids Struc.* **3**, 445 (1967).
22. J. L. Sewall and E. C. Naumann, An experimental and analytical vibration study of thin cylindrical shells with and without longitudinal stiffeners. *NASA TN D*-4705 (1968).
23. A. Rosen and J. Singer, Vibrations of axially loaded stiffened cylindrical shells: Part I—Theoretical Analysis. *TAE Report No.* 162, Technion-Israel Institute of Technology, Dept. of Aeronautical Engineering, Haifa, Israel (Feb. 1974).
24. A. Rosen and J. Singer, Vibrations of axially loaded stiffened cylindrical shells: Part II—Experimental Analysis. *TAE Report No.* 163, Technion-Israel Institute of Technology, Dept. of Aeronautical Engineering, Haifa, Israel (Aug. 1973).
25. A. Rosen and J. Singer, Vibrations of axially loaded stiffened cylindrical shells. *J. of Sound and Vibration* **34** (3), 357 (1974) (condensed from Refs. [23 and 24]).
26. M. M. Mikulas and J. A. McElman, On the free vibrations of eccentrically stiffened cylindrical shells and flat plates. *NASA TN D*-3010 (Sept. 1965).
27. J. A. McElman, M. M. Mikulas and M. Stein, Static and dynamic effects of eccentric stiffening of plates and cylindrical shells. *AIAA J.* **4**, 887 (1966).
28. S. Parthan and D. J. Johns, Vibrations and flutter of unstiffened and orthogonally stiffened circular cylindrical shells. *TT.* 7106, Vol. I, II. Dept. of Transport Tech., Loughborough Univ. of Technology (June 1971).
29. S. Parthan and D. J. John, Effects of inplane and rotary inertia on the frequencies of eccentrically stiffened cylindrical shells. *AIAA J.* **8**, 592 (1970).

30. A. Harari and M. L. Baron, *Discrete analysis for the dynamic response of stiffened shells.* Paul Weidlinger, Consulting Engineer, Tech. Report No. 10, Office of Naval Research, prepared under contract No. N00014-70-C-0359, Task No. NR-064-464 (March 1971).
31. A. Harari and M. L. Baron, Analysis for the dynamic response of stiffened shells. *J. of Appl. Mech., ASME* **40**, 1085 (1973).
32. T. Weller and J. Singer, Experimental studies on the buckling of 7075-T6 aluminum alloy integrally stringer-stiffened shells. *TAE Report No.* 135. Technion Research and Development Foundation, Haifa, Israel (Oct. 1971).
33. A. Rosen and J. Singer, Vibrations of axially loaded stiffened cylindrical shells with elastic restraints. *TAE Report No.* 208, Technion-Israel Institute of Technology, Dept. of Aeronautical Engineering (Jan., 1975).
34. A. Rosen and J. Singer, Effect of axisymmetric imperfections on the vibrations of cylindrical shells under axial compression. *AIAA J.* **12**, 995 (1947).
35. A. Rosen and J. Singer, Influence of asymmetric imperfections on the vibrations of axially compressed cylindrical shells. *TAE Report No.* 212. Technion Research and Development Foundation, Haifa, Israel (March 1975).
36. A. Rosen and J. Singer, Vibrations and buckling of eccentrically loaded stiffened cylindrical shells. *Experimental Mechanics* **16**, 88 1976. Also *TAE Report No.* 205, Technion Research and Development Foundation, Haifa, Israel (June, 1974).
37. J. Singer and A. Rosen, Design criteria for buckling and vibration of imperfect stiffened shells. *ICAS Paper No.* 74-06. The 9th Congress of the International Council of the Aeronautical Sciences, Haifa Auditorium, Haifa, Israel, 25–30 August, 1974.

APPENDIX

If one assumes a membrane prestress state, and u, v, w represent the mode of vibration, one obtains the three Flügge type equations of equilibrium:

$$\left[1+\mu_1+\underline{\frac{N_x(1-\nu^2)}{Eh}}\right]u_{,xx}+\left[\frac{1-\nu}{2}(1+\underline{T})-\underline{\frac{pR(1-\nu^2)}{Eh}}\right]u_{,yy}+\frac{1+\nu}{2}v_{,xy}$$
$$-\left[\nu+\underline{\frac{pR(1-\nu^2)}{Eh}}\right]w_{,x}-(\chi_1-\underline{T})w_{,xxx}-\underline{T\frac{1-\nu}{2}}w_{,xyy}=\frac{R^2\bar{M}(1-\nu^2)}{Eh}\ddot{u}$$

$$\frac{1+\nu}{2}u_{,xy}+\left[1+\mu_2-\underline{\frac{pR(1-\nu^2)}{Eh}}\right]v_{,yy}+\left[\frac{1-\nu}{2}(1+\underline{3T})+\underline{T\eta_{t1}}+\underline{\frac{\bar{N}_x(1-\nu^2)}{Eh}}\right]v_{,xx}$$
$$-\left[1+\mu_2+\underline{\chi_2}+\underline{T\eta_{02}}-\underline{\frac{pR(1-\nu^2)}{Eh}}\right]w_{,y}-(\chi_2+\underline{T\eta_{02}})w_{,yyy}+\underline{T\left(\frac{3-\nu}{2}+\eta_{t1}\right)}w_{,xxy}=\frac{R^2\bar{M}(1-\nu^2)}{Eh}\ddot{v}$$

$$\left[\nu+\underline{\frac{pR(1-\nu^2)}{Eh}}\right]u_{,x}+(\chi_1-\underline{T})u_{,xxx}+\underline{T\frac{1-\nu}{2}u_{,xyy}}+\left[1+\mu_2+\underline{\chi_2+T\eta_{02}-\frac{pR(1-\nu^2)}{Eh}}\right]v_{,y}+(\chi_2+\underline{T\eta_{02}})v_{,yyy}$$
$$\underline{-T\left(\frac{3-\nu}{2}+\eta_{t1}\right)v_{,xxy}}-(1+\mu_2+\underline{2\chi_2})w-\left[2\chi_2+\frac{pR(1-\nu^2)}{Eh}\right]w_{,yy}+\frac{\bar{N}_x(1-\nu^2)}{Eh}w_{,xx}$$
$$\underline{-T[(1+3\eta_{02})w}+\underline{2(1+2\eta_{02})w_{,yy}}+(1+\eta_{01})w_{,xxxx}$$
$$+(2+\eta_{t1}+\eta_{t2})w_{,xxyy}+(1+\eta_{02})w_{,yyyy}]=\frac{R^2\bar{M}(1-\nu^2)}{Eh}\ddot{w} \qquad \text{(A1)}$$

where the terms underlined are those which do not appear in Donnell's theory.

The boundary conditions are

(a) $$(1+\mu_1)u_{,x}+\nu(v_{,y}-w)-\chi_1w_{,xx}+\underline{Tw_{,xx}}+(-1)^rk_1{}^ru=0$$

(b) $$\left[\frac{1-\nu}{2}+\underline{\frac{\bar{N}_x(1-\nu^2)}{Eh}}\right]u_{,y}+\frac{1-\nu}{2}(1+\underline{3T})v_{,x}+\underline{T\eta_{t1}v_{,x}}+\underline{3T\frac{1-\nu}{2}w_{,xy}}+\underline{T\eta_{t1}w_{,xy}}+(-1)^rk_2{}^rv=0$$

(c) $$\left(\underline{1}+\frac{12R^2}{h^2}\chi_1\right)u_{,xx}-\underline{\frac{1-\nu}{2}u_{,yy}}+\underline{\left(\frac{3-\nu}{2}+\eta_{t1}\right)v_{,xy}}+(1+\eta_{01})w_{,xxx}$$
$$+(2-\nu+\eta_{t1}+\eta_{t2})w_{,xyy}-\frac{\bar{N}_x\cdot 12\cdot(1-\nu^2)R^2}{Eh^3}w_{,x}-(-1)^rk_3{}^rw=0$$

(d) $$\left(\underline{1}-\frac{12R^2}{h_2}\chi_1\right)u_{,x}+\underline{\nu v_{,y}}+\underline{\nu w_{,yy}}+(1+\eta_{01})w_{,xx}+(-1)^rk_4{}^rw_{,x}=0 \qquad \text{(A2)}$$

where $k_1{}^r$, $k_2{}^r$, $k_3{}^r$ and $k_4{}^r$ are the nondimensional elastic restraints defined in the Notation.